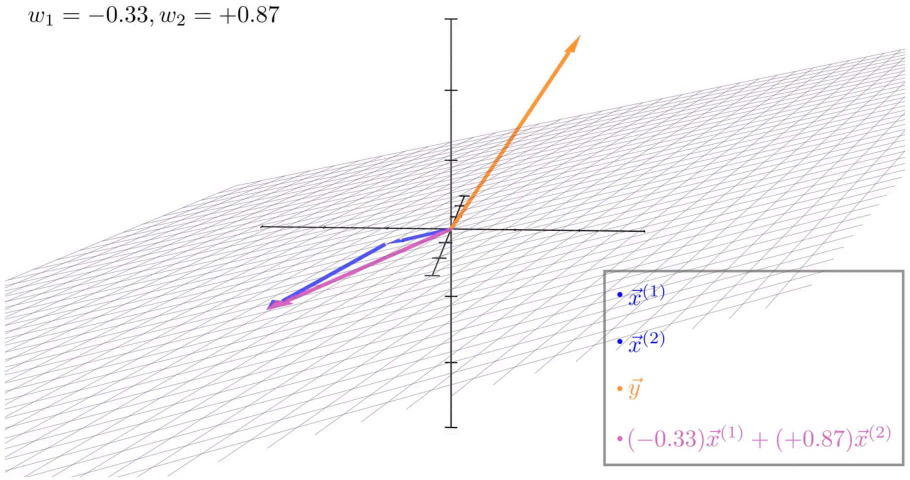
Moving to multiple dimensions

- Let's now consider three vectors, $\vec{y}$, $\vec{x}^{(1)}$, and $\vec{x}^{(2)}$, all in $\mathbb{R}^n$.
- **Question**: What vector in $\text{span}(\vec{x}^{(1)}, \vec{x}^{(2)})$ is closest to $\vec{y}$?
  - Vectors in $\text{span}(\vec{x}^{(1)}, \vec{x}^{(2)})$ are of the form $w_1\vec{x}^{(1)} + w_2\vec{x}^{(2)}$, where $w_1, w_2 \in \mathbb{R}$ are scalars.
- Before trying to answer, let's watch 🎥 **this animation that Jack, one of our tutors, made**.



Minimizing projection error in multiple dimensions

- **Question**: What vector in $\text{span}(\vec{x}^{(1)}, \vec{x}^{(2)})$ is closest to $\vec{y}$?
    - That is, what vector minimizes $\|\vec{e}\|$, where:

$$\vec{e} = \vec{y} - w_1 \vec{x}^{(1)} - w_2 \vec{x}^{(2)}$$

- **Answer**: It's the vector such that $w_1 \vec{x}^{(1)} + w_2 \vec{x}^{(2)}$ is **orthogonal** to $\vec{e}$.
- **Issue**: Solving for w_1 and w_2 in the following equation is difficult:

$$\underbrace{\left(w_1 \vec{x}^{(1)} + w_2 \vec{x}^{(2)}\right)}_{\text{any vector in span}(\vec{x}^{(1)}, \vec{x}^{(2)}) \text{ can be written in this form!}} \cdot \underbrace{\left(\vec{y} - w_1 \vec{x}^{(1)} - w_2 \vec{x}^{(2)}\right)}_{\vec{e}} = 0$$

Minimizing projection error in multiple dimensions

- It's hard for us to solve for w_1 and w_2 in:

$$\left(w_1\vec{x}^{(1)} + w_2\vec{x}^{(2)}\right) \cdot \underbrace{\left(\vec{y} - w_1\vec{x}^{(1)} - w_2\vec{x}^{(2)}\right)}_{\vec{e}} = 0$$

- **Observation**: All we really need is for $\vec{x}^{(1)}$ and $\vec{x}^{(2)}$ to individually be orthogonal to $\vec{e}$.
  - That is, it's sufficient for $\vec{e}$ to be orthogonal to the spanning vectors themselves.
- If $\vec{x}^{(1)} \cdot \vec{e} = 0$ and $\vec{x}^{(2)} \cdot \vec{e} = 0$, then:

$$\begin{aligned}\left(w_1\vec{x}^{(1)} + w_2\vec{x}^{(2)}\right) \cdot \vec{e} &= w_1\vec{x}^{(1)} \cdot \vec{e} + w_2\vec{x}^{(2)} \cdot \vec{e} \\ &= w_1\left(\vec{x}^{(1)} \cdot \vec{e}\right) + w_2\left(\vec{x}^{(2)} \cdot \vec{e}\right) \\ &= w_1(0) + w_2(0) \\ &= \boxed{0}\end{aligned}$$

Minimizing projection error in multiple dimensions

- **Question**: What vector in $\text{span}(\vec{x}^{(1)}, \vec{x}^{(2)})$ is closest to $\vec{y}$?
- **Answer**: It's the vector such that $w_1\vec{x}^{(1)} + w_2\vec{x}^{(2)}$ is **orthogonal** to $\vec{e} = \vec{y} - w_1\vec{x}^{(1)} - w_2\vec{x}^{(2)}$.
- **Equivalently, it's the vector such that $\vec{x}^{(1)}$ and $\vec{x}^{(2)}$ are both orthogonal to $\vec{e}$:**

$$\boxed{\begin{aligned} \vec{x}^{(1)} \cdot \left(\vec{y} - w_1\vec{x}^{(1)} - w_2\vec{x}^{(2)}\right) &= 0 \\ \vec{x}^{(2)} \cdot \underbrace{\left(\vec{y} - w_1\vec{x}^{(1)} - w_2\vec{x}^{(2)}\right)}_{\vec{e}} &= 0 \end{aligned}}$$

need to find w_1^*, w_2^*!

- This is a system of two equations, two unknowns (w_1 and w_2), but it still looks difficult to solve.

Now what?

- We're looking for the scalars w_1 and w_2 that satisfy the following equations:

$$\begin{aligned} \vec{x}^{(1)} \cdot \left(\vec{y} - w_1 \vec{x}^{(1)} - w_2 \vec{x}^{(2)}\right) &= 0 \\ \vec{x}^{(2)} \cdot \underbrace{\left(\vec{y} - w_1 \vec{x}^{(1)} - w_2 \vec{x}^{(2)}\right)}_{\vec{e}} &= 0 \end{aligned}$$

- In this example, we just have two spanning vectors, $\vec{x}^{(1)}$ and $\vec{x}^{(2)}$.
- If we had any more, this system of equations would get extremely messy, extremely quickly.
- **Idea**: Rewrite the above system of equations **as a single equation, involving matrix-vector products**.