$\sqrt{20}\sqrt{34}\cos\theta = 22 \Rightarrow \cos\theta = \dfrac{22}{\sqrt{20}\sqrt{34}}$

$\cos\theta = \dfrac{\cancel{22}^{\,11}}{\cancel{2}\sqrt{5}\sqrt{34}}$

Dot product: geometric definition

- The computational definition of the dot product:

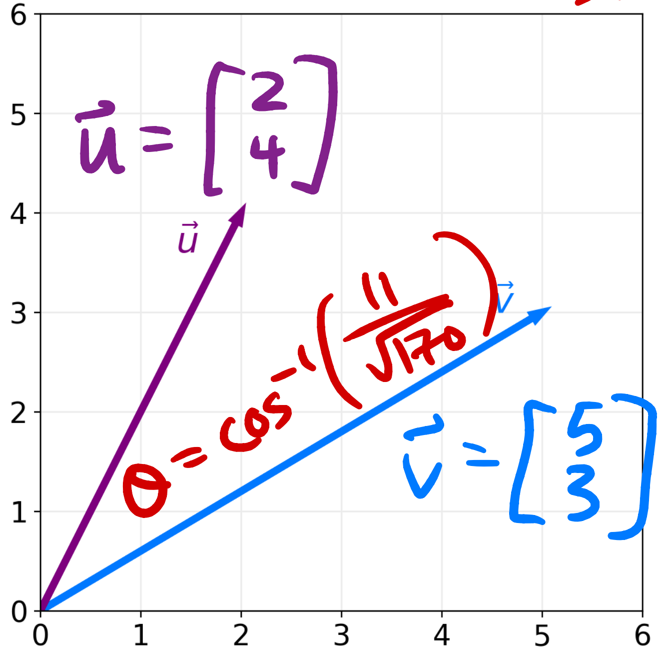
$$\vec{u}\cdot\vec{v} = \sum_{i=1}^{n} u_i v_i = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n$$

- The geometric definition of the dot product:

$$\vec{u}\cdot\vec{v} = \|\vec{u}\|\|\vec{v}\|\cos\theta$$

where θ is the angle between $\vec{u}$ and $\vec{v}$.

- The two definitions are equivalent! This equivalence allows us to find the angle θ between two vectors.



equal!

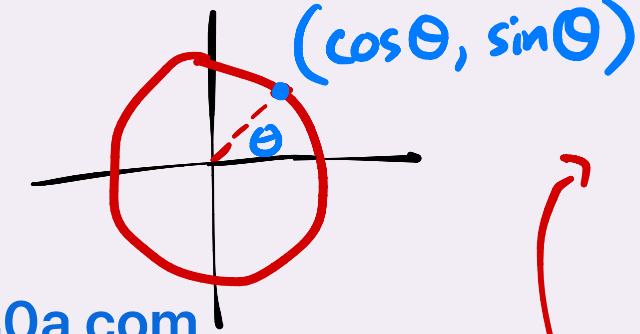
$\vec{u}\cdot\vec{v} = (2)(5) + (4)(3) = 22$

$\vec{u}\cdot\vec{v} = \|\vec{u}\|\,\|\vec{v}\|\cos\theta = \sqrt{2^2+4^2}\,\sqrt{5^2+3^2}\cos\theta = \sqrt{20}\sqrt{34}\cos\theta$

Question 🤔

Answer at q.dsc40a.com

What is the value of θ in the plot to the right?



$\vec{u} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} -2 \\ -4 \end{bmatrix}$

① $\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v} = \begin{bmatrix} 4 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ -4 \end{bmatrix}$

$= 4(-2) + 1(-4) = -12$

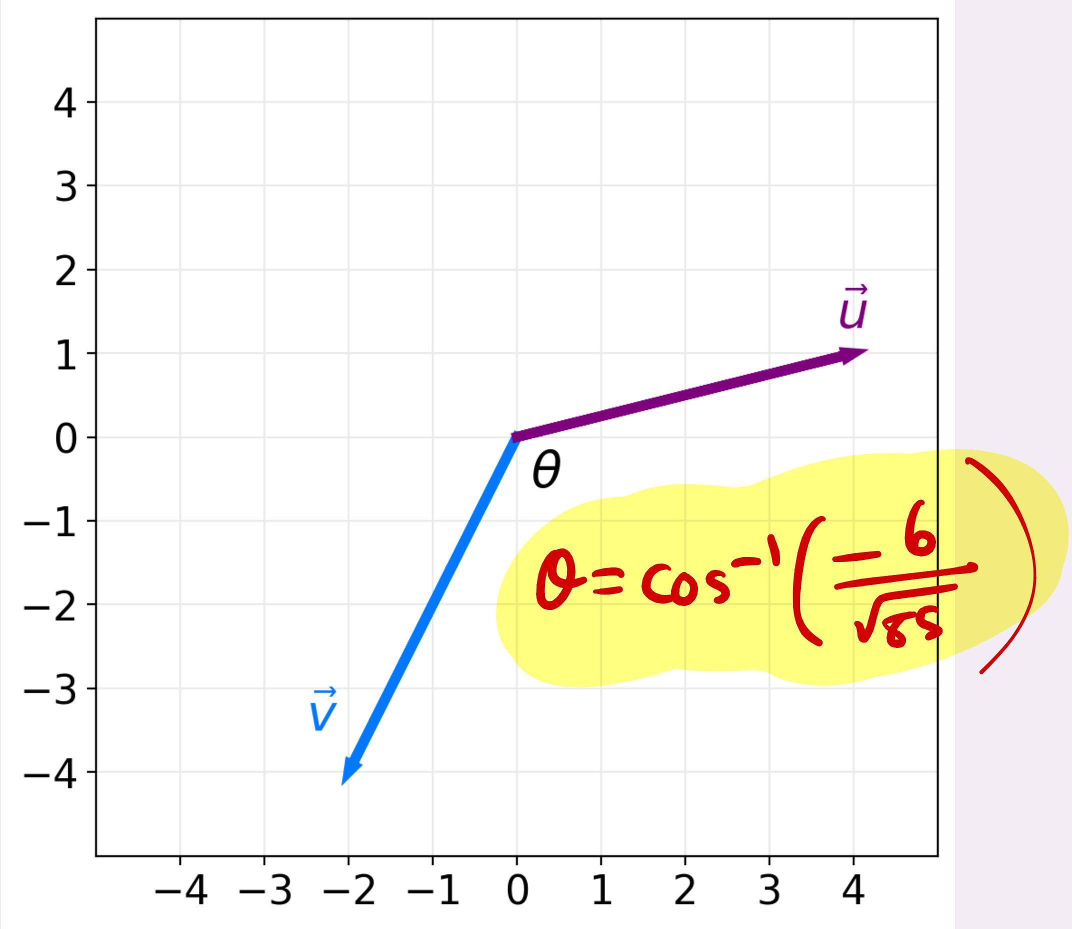
② $\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta$

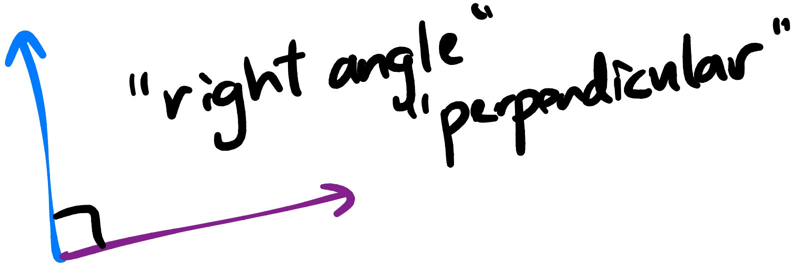
$= \sqrt{4^2 + 1^2} \sqrt{(-2)^2 + (-4)^2} \cos\theta$

$= \sqrt{17} \sqrt{20} \cos\theta$

$\sqrt{17}\sqrt{20} \cos\theta = -12$

$\Rightarrow \cos\theta = \frac{-12}{\sqrt{17}\sqrt{20}} = \frac{-6}{\sqrt{17}\sqrt{5}} = \frac{-6}{\sqrt{85}}$





Orthogonal vectors

- Recall: $\cos 90^{\mathrm{o}} = 0$.
- Since $\vec{u} \cdot \vec{v} = ||\vec{u}|| \, ||\vec{v}|| \cos \theta$, if the angle between two vectors is 90^{o}, their dot product is $||\vec{u}|| \, ||\vec{v}|| \cos 90^{\mathrm{o}} = 0$.
- If the angle between two vectors is 90^{o}, we say they are perpendicular, or more generally, **orthogonal**.
- **Key idea**:

$$\boxed{\text{two vectors are } \textbf{orthogonal} \iff \vec{u} \cdot \vec{v} = 0}$$

"if and only if"
bidirectional statement

Exercise

Find a non-zero vector in $\mathbb{R}^3$ orthogonal to:

$$\vec{v} = \begin{bmatrix} 2 \\ 5 \\ -8 \end{bmatrix}$$

could find solutions to

$2u_1 + 5u_2 - 8u_3 = 0$

Infinitely many possibilities!

$\begin{bmatrix} 2 \\ 12 \\ 8 \end{bmatrix}$: $(2)(2) + (12)(5) + (8)(-8)$

$= 4 + 60 - 64$

$= 0$

$\begin{bmatrix} 0 \\ 8 \\ 5 \end{bmatrix}$: $(0)(2) + (8)(5) + (5)(-8)$

$= 40 - 40$

$= 0$