Matrices

Matrices

- An $n \times d$ **matrix** is a table of numbers with n rows and d columns.
- We use upper-case letters to denote matrices.

$$A = \begin{bmatrix} 2 & 5 & 8 \\ -1 & 5 & -3 \end{bmatrix} \quad 2 \times 3$$

- Since A has two rows and three columns, we say $A \in \mathbb{R}^{2 \times 3}$.

the set of matrices with 2 rows and 3 columns

- **Key idea**: Think of a matrix as **several column vectors, stacked next to each other**.

$$\begin{bmatrix} 2 \\ -1 \end{bmatrix} \quad \begin{bmatrix} 5 \\ 5 \end{bmatrix} \quad \begin{bmatrix} 8 \\ -3 \end{bmatrix}$$

Matrix addition and scalar multiplication

- We can add two matrices only if they have the same dimensions.
- Addition occurs elementwise:

$$\begin{bmatrix} 2 & 5 & 8 \\ -1 & 5 & -3 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 7 & 11 \\ -1 & 6 & -1 \end{bmatrix}$$

- Scalar multiplication occurs elementwise, too:

$$2\begin{bmatrix} 2 & 5 & 8 \\ -1 & 5 & -3 \end{bmatrix} = \begin{bmatrix} 4 & 10 & 16 \\ -2 & 10 & -6 \end{bmatrix}$$

Matrix-matrix multiplication

- **Key idea**: We can multiply matrices A and B **if and only if**:

$$\boxed{\# \text{ columns in } A = \# \text{ rows in } B}$$

- If A is $n \times d$ and B is $d \times p$, then AB is $n \times p$.
- Example: If A is as defined below, what is $A^T A$?

$$A = \begin{bmatrix} 2 & 5 & 8 \\ -1 & 5 & -3 \end{bmatrix}_{2\times 3}$$

$$A^T = \begin{bmatrix} 2 & -1 \\ 5 & 5 \\ 8 & -3 \end{bmatrix}_{3\times 2}$$

$$A^T A = \begin{bmatrix} 5 & & \\ 5 & & \boxed{25} \\ 19 & & \end{bmatrix}_{3\times 3}$$

Question 🤔

Answer at **q.dsc40a.com**

Assume A, B, and C are all matrices. Select the **incorrect** statement below.

- A. $A(B+C) = AB + AC$.
- B. $A(BC) = (AB)C$.
- C. $AB = BA$.
- D. $(A+B)^T = A^T + B^T$.
- E. $(AB)^T = B^T A^T$.

$A_{5\times 7} B_{7\times 5} \rightarrow 5 \times 5$

different dimensions!

$B_{7\times 5} A_{5\times 7} \rightarrow 7 \times 7$

Matrix-vector multiplication

- A vector $\vec{v} \in \mathbb{R}^n$ is a matrix with n rows and 1 column.

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

- Suppose $A \in \mathbb{R}^{n \times d}$.
  - What must the dimensions of $\vec{v}$ be in order for the product $A\vec{v}$ to be valid?

    $A_{n \times d} \vec{V}_{d \times 1} \Rightarrow \vec{v} \in \mathbb{R}^d$ d components

  - What must the dimensions of $\vec{v}$ be in order for the product $\vec{v}^T A$ to be valid?

    $\vec{V}^T_{1 \times n} A_{n \times d} \Rightarrow \vec{v} \in \mathbb{R}^n$ n components

One view of matrix-vector multiplication

- One way of thinking about the product $A\vec{v}$ is that it is **the dot product of $\vec{v}$ with every row of A**.
- Example: What is $A\vec{v}$?

$2(2)+(-1)(5)+(-5)(8)$
$=4-5-40=-41$

$$A = \begin{bmatrix} 2 & 5 & 8 \\ -1 & 5 & -3 \end{bmatrix}_{2\times 3} \qquad \vec{v} = \begin{bmatrix} 2 \\ -1 \\ -5 \end{bmatrix}_{3\times 1}$$

$$A\vec{v} = \begin{bmatrix} -41 \\ 8 \end{bmatrix}$$

$\vec{v} \in \mathbb{R}^3$

$A\vec{v} \in \mathbb{R}^2$

$2(-1)+(-1)(5)+(-5)(-3)$
$=-2-5+15=8$

Another view of matrix-vector multiplication

- Another way of thinking about the product $A\vec{v}$ is that it is **a linear combination of the columns of A, using the weights in $\vec{v}$.**
- Example: What is $A\vec{v}$?

$$A = \begin{bmatrix} 2 & 5 & 8 \\ -1 & 5 & -3 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} 2 \\ -1 \\ -5 \end{bmatrix}$$

$$A\vec{v} = \underbrace{2\begin{bmatrix} 2 \\ -1 \end{bmatrix} + (-1)\begin{bmatrix} 5 \\ 5 \end{bmatrix} + (-5)\begin{bmatrix} 8 \\ -3 \end{bmatrix}}_{\text{a linear combination of the columns of } A!} = \begin{bmatrix} -41 \\ 8 \end{bmatrix}$$

Matrix-vector products create linear combinations of columns!

- **Key idea**: It'll be very useful to think of the matrix-vector product $A\vec{v}$ as a linear combination of the columns of A, using the weights in $\vec{v}$.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1d} \\ a_{21} & a_{22} & \cdots & a_{2d} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nd} \end{bmatrix}_{n \times d} \qquad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_d \end{bmatrix}_{d \times 1}$$

$$\downarrow$$

$$\boxed{A\vec{v} = v_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix} + v_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{bmatrix} + \ldots + v_d \begin{bmatrix} a_{1d} \\ a_{2d} \\ \vdots \\ a_{nd} \end{bmatrix}}$$

$\Rightarrow$ result is a vector in $\mathbb{R}^n$!