# Spans and projections

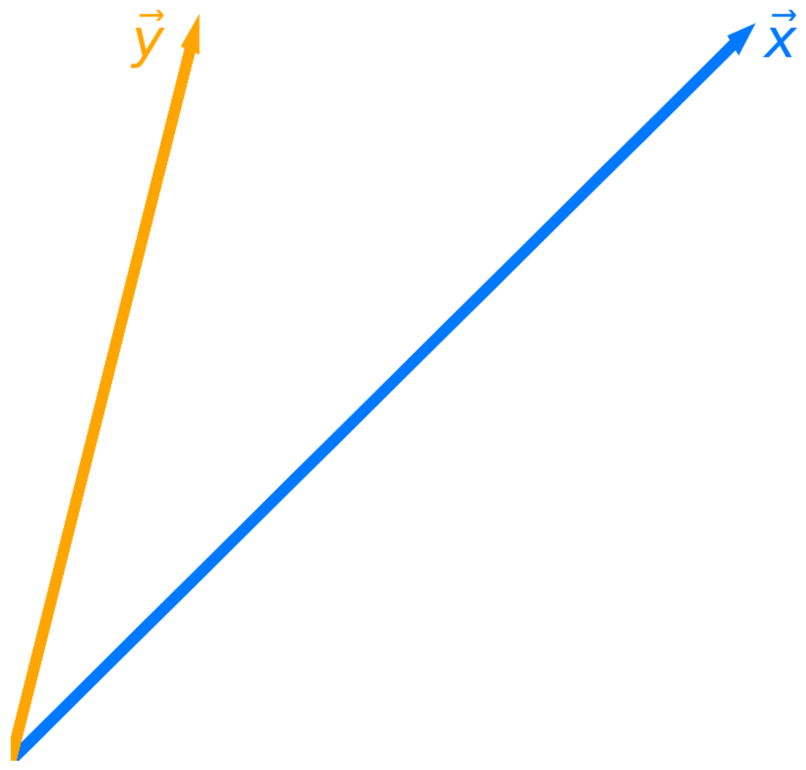
# Projecting onto a single vector

- Let $\vec{x}$ and $\vec{y}$ be two vectors in $\mathbb{R}^n$.
- The span of $\vec{x}$ is the set of all vectors of the form:

$$w\vec{x}$$

where $w \in \mathbb{R}$ is a scalar.

- **Question**: What vector in $\text{span}(\vec{x})$ is closest to $\vec{y}$?
- The vector in $\text{span}(\vec{x})$ that is closest to $\vec{y}$ is the ____________ **projection of $\vec{y}$ onto $\text{span}(\vec{x})$**.



## Projection error

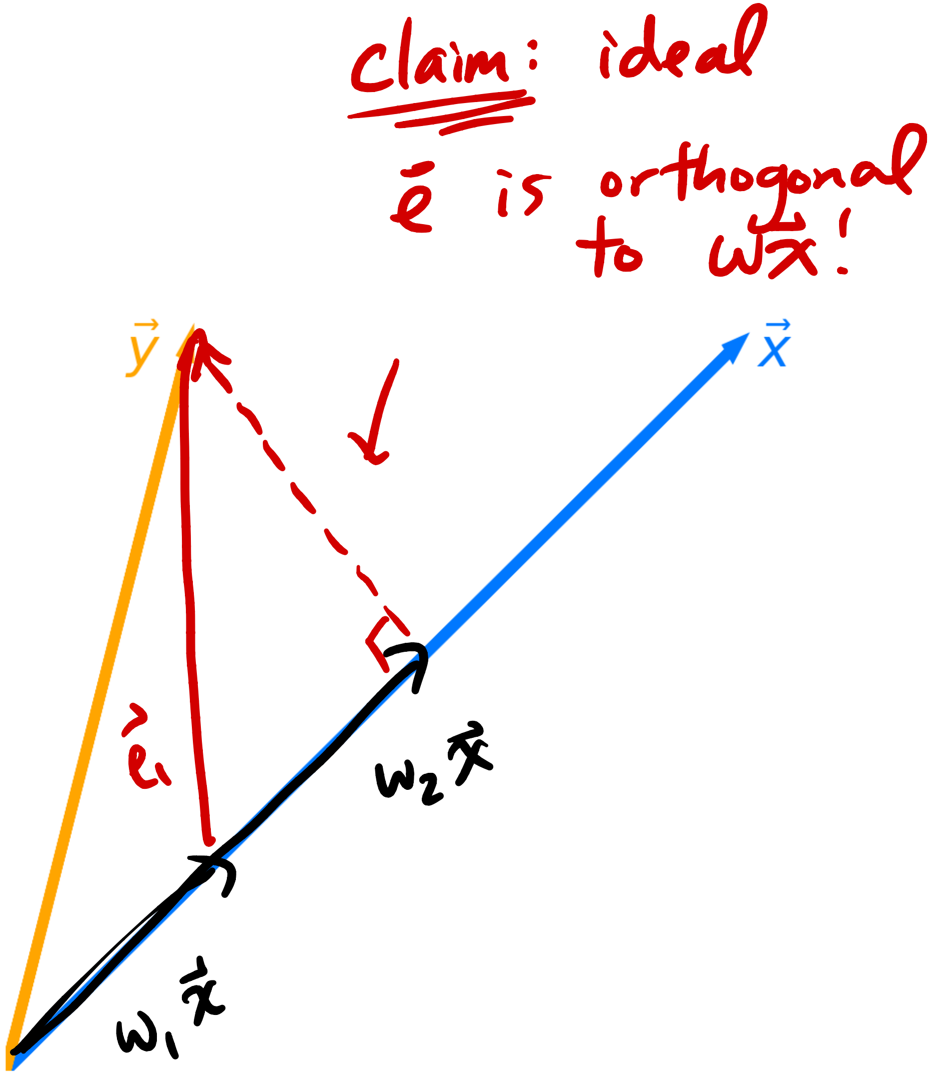
- Let $\vec{e} = \vec{y} - w\vec{x}$ be the **projection error**: that is, the vector that connects $\vec{y}$ to $\text{span}(\vec{x})$.
- **Goal**: **Find the $w$ that makes $\vec{e}$ as short as possible.**
    - That is, minimize:

$$||\vec{e}||$$

    - Equivalently, minimize:

$$||\vec{y} - w\vec{x}||$$

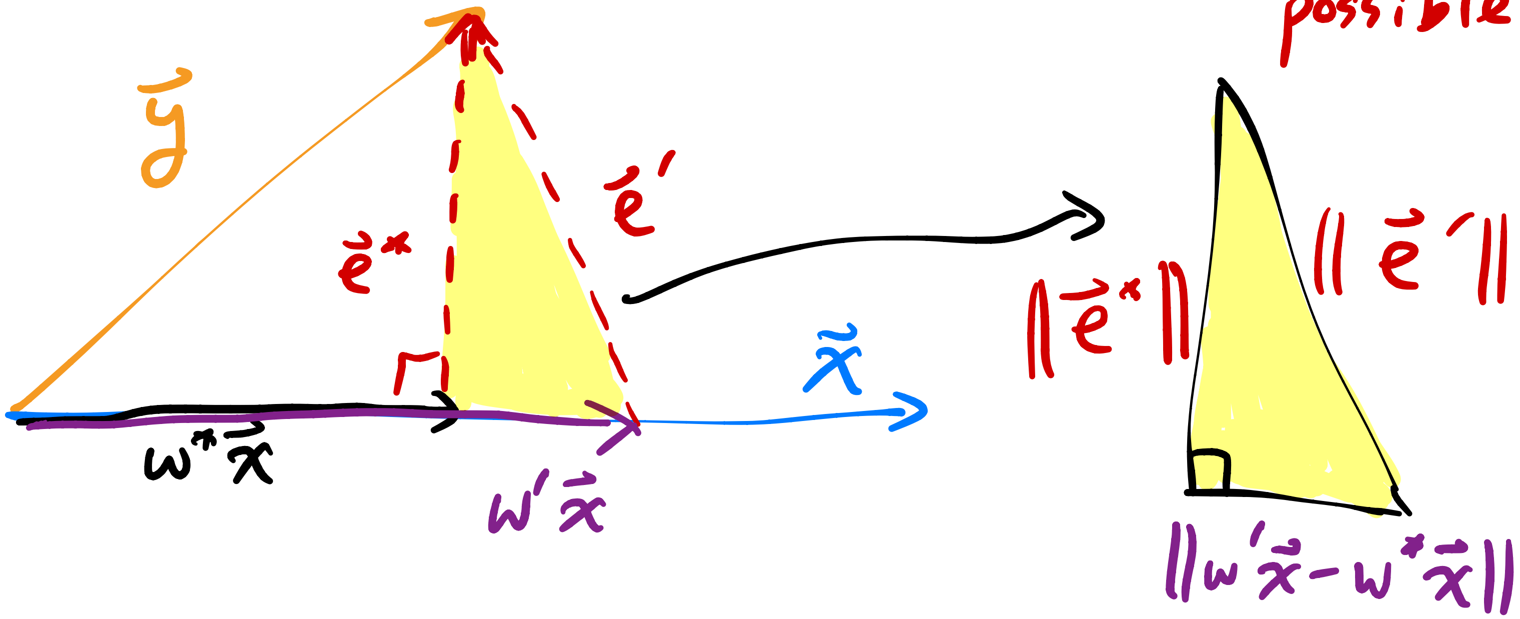
- **Idea**: To make $\vec{e}$ has short as possible, it should be **orthogonal to $w\vec{x}$**.



## Minimizing projection error

- **Goal: Find the $w$ that makes $\vec{e} = \vec{y} - w\vec{x}$ as short as possible.**
- **Idea**: To make $\vec{e}$ as short as possible, it should be **orthogonal to $w\vec{x}$**.
- Can we prove that making $\vec{e}$ orthogonal to $w\vec{x}$ minimizes $\|\vec{e}\|$?

<u>Goal</u>: Prove that $\vec{e}^*$ is the shortest possible error vector.




Pythagorean theorem:

$$\|\vec{e}'\|^2 = \|\vec{e}^*\|^2 + \underbrace{\|w'\vec{x} - w^*\vec{x}\|^2}_{\geq 0}$$

$$\|\vec{e}'\|^2 \geq \|\vec{e}^*\|^2$$

$\Rightarrow$ $\vec{e}^*$ is the shortest possible error vector!

# Minimizing projection error

- **Goal: Find the $w$ that makes $\vec{e} = \vec{y} - w\vec{x}$ as short as possible.**
- Now we know that to minimize $||\vec{e}||$, $\vec{e}$ must be orthogonal to $w\vec{x}$.
- Given this fact, how can we solve for $w$?

$\vec{e}$ orthogonal to $w\vec{x}$ $\Rightarrow$ $\boxed{w\vec{x} \cdot \vec{e} = 0}$

$$w\vec{x} \cdot (\vec{y} - w\vec{x}) = 0$$

$$\vec{x} \cdot (\vec{y} - w\vec{x}) = 0$$

$$\vec{x} \cdot \vec{y} - \vec{x} \cdot (w\vec{x}) = 0$$

$$\vec{x} \cdot \vec{y} - w(\vec{x} \cdot \vec{x}) = 0$$

$$\vec{x} \cdot \vec{y} = w(\vec{x} \cdot \vec{x})$$

$$\Rightarrow \boxed{w^* = \frac{\vec{x} \cdot \vec{y}}{\vec{x} \cdot \vec{x}}}$$

The $w$ that makes the error vector as short as possible!!!

# Orthogonal projection

- **Question**: What vector in $\text{span}(\vec{x})$ is closest to $\vec{y}$?
- **Answer**: It is the vector $w^* \vec{x}$, where:

$$w^* = \frac{\vec{x} \cdot \vec{y}}{\vec{x} \cdot \vec{x}}$$

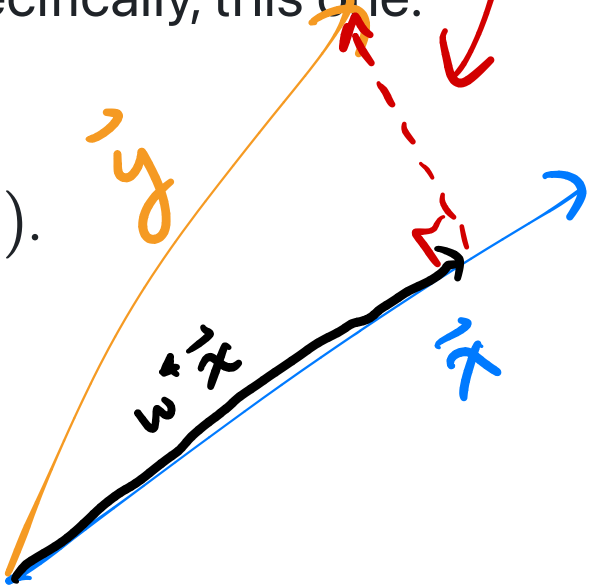
- Note that $w^*$ is the solution to a minimization problem, specifically, this one:

$$\text{error}(w) = \|\vec{e}\| = \|\vec{y} - w\vec{x}\|$$

- We call $w^* \vec{x}$ the **orthogonal projection of $\vec{y}$ onto $\text{span}(\vec{x})$**.
    - Think of $w^* \vec{x}$ as the "shadow" of $\vec{y}$.

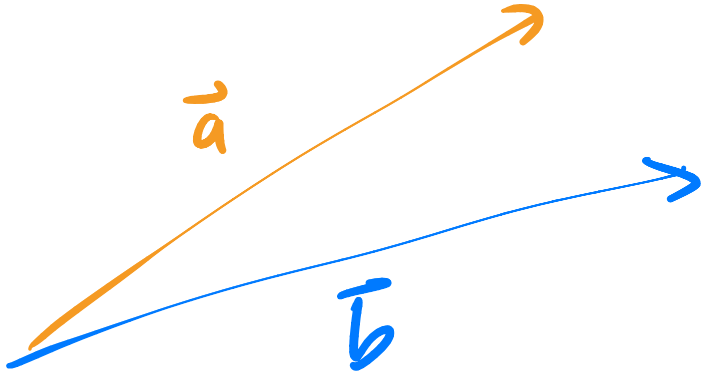
$\vec{e}$ is orthogonal to $\vec{x}$!

minimum possible $\|\vec{e}\|$



## Exercise

Let $\vec{a} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} -1 \\ 9 \end{bmatrix}$.




What is the orthogonal projection of $\vec{a}$ onto $\text{span}(\vec{b})$?

Your answer should be of the form $w^* \vec{b}$, where $w^*$ is a scalar.

$$w^* = \frac{\vec{b} \cdot \vec{a}}{\vec{b} \cdot \vec{b}} = \frac{(-1)(5) + (9)(2)}{(-1)^2 + (9)^2} = \boxed{\frac{13}{82}}$$

orthogonal projection of $\vec{a}$ onto $\text{span}(\vec{b})$ is $\boxed{\frac{13}{82} \vec{b}}$.